Inequality 9

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Q> If $a, b, c > 0$ and $abc = 1$ the prove that,

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} \geq 3$$

Ans:- $\frac{1+ab}{1+a} = \frac{1+\frac{1}{c}}{1+a} = \frac{c+1}{c(1+a)}$

$\frac{1+bc}{1+b} = \frac{1+a}{a(1+b)}$ $\qquad$ $\frac{1+ca}{1+c} = \frac{1+b}{b(1+c)}$

$\Rightarrow \frac{1+a}{a(1+b)} + \frac{1+b}{b(1+c)} + \frac{1+c}{c(1+a)} \geq 3\sqrt[3]{\frac{(1+a)(1+b)(1+c)}{(1+a)(1+b)(1+c)\,abc}} = 3$

Q> Let us take two sequences $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ and one permutation $(z_1, z_2, \ldots, z_n)$ of $(y_1, y_2, \ldots, y_n)$ Prove that,

$$\underbrace{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}_{S_1} \leq \underbrace{(x_1-z_1)^2 + (x_2-z_2)^2 + \cdots + (x_n-z_n)^2}_{S_2}$$

Ans:- $S_1 - S_2 = \sum_{i=1}^{n}(x_i-y_i)^2 - \sum_{i=1}^{n}(x_i-z_i)^2$

$= \sum_{i=1}^{n}(\cancel{x_i^2} - 2x_iy_i + y_i^2) - \sum_{i=1}^{n}(\cancel{x_i^2} - 2x_iz_i + z_i^2)$

$= \sum_{i=1}^{n}(y_i^2 - z_i^2) + \sum_{i=1}^{n}(2x_iz_i - 2x_iy_i)$

$= \sum_{i=1}^{n}(y_i^2) - \sum_{i=1}^{n}(z_i)^2 + 2\sum_{i=1}^{n}(x_iz_i) - 2\sum_{i=1}^{n}(x_iy_i)$

$y_i = z_j$

$y_i^2 - z_j^2 = 0$

$= \sum_{i=1}^{n}(y_i^2) - \sum_{i=1}^{n}(y_i^2) + B$

$= 0 + B = B$

By rearrangement inequality $\sum x_iz_i \leq \sum x_iy_i \Rightarrow B < 0$

$\Rightarrow S_1 - S_2 \leq 0$

$\Rightarrow S_1 \leq S_2$

Q> Let $x_1, x_2, \ldots, x_n$ be distinct positive integers, then prove that
$$\frac{x_1}{1^2} + \frac{x_2}{2^2} + \cdots + \frac{x_n}{n^2} \geq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Ans:– $1 \leq 2 \leq \cdots \leq n \Rightarrow \frac{1}{n} \leq \frac{1}{n-1} \leq \cdots \leq \frac{1}{1} \Rightarrow \frac{1}{n^2} \leq \frac{1}{(n-1)^2} \leq \cdots \leq \frac{1}{1^2}$

$y_1 \leq y_2 \leq \cdots \leq y_n$ and $(y_1, y_2, \ldots, y_n)$ is a permutation $(x_1, x_2, \ldots, x_n)$

$$\sum_{i=1}^{n} \frac{x_i}{i^2} \geq \sum_{i=1}^{n} \frac{y_i}{(n-i+1)^2} \longrightarrow \frac{y_1}{1^2} + \frac{y_2}{2^2} + \cdots + \frac{y_n}{n^2}$$

$$\Rightarrow \sum_{i=1}^{n} \frac{x_i}{i^2} \geq \sum_{i=1}^{n} \frac{y_i}{(n-i+1)^2} \geq \sum_{i=1}^{n} \frac{n-i+1}{(n-i+1)^2} = \sum_{i=1}^{n} \frac{1}{n-i+1} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

Suppose y contain some element $y_k' > n$ and $y_k \leq n$ then we get,

$$S_1 = \frac{y_1}{1^2} + \frac{y_2}{2^2} + \cdots + \frac{y_k'}{k^2} + \cdots + \frac{y_n}{n^2}$$

$$S_2 = \frac{y_1}{1^2} + \frac{y_2}{2^2} + \cdots + \frac{y_k}{k^2} + \cdots + \frac{y_n}{n^2}$$

$$S_1 - S_2 = \frac{y_k'}{k^2} - \frac{y_k}{k^2} > 0 \quad \text{as } y_k' > n \text{ and } y_k \leq n$$

$\Rightarrow S_1 > S_2$

$\Rightarrow$ Smallest such S will be from all y_i's $\leq n$

So $y = \{1, 2, \ldots, n\}$

$$\{n-i+1\}_{i=1}^{n} = \{i\}_{i=1}^{n}$$

$n-1+1 = n$

$n-2+1 = n-1$

$n-3+1 = n-2 \ldots$

Q> Suppose a, b, c be the lengths of the sides of a triangle. Prove that $a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c) \leq 3abc$

Ans:– $c \leq b \leq a$

$$a(b+c-a) \leq b(a+c-b) \leq c(a+b-c)$$

Ans:- $c \leq b \leq a$

$$a(b+c-a) \leq b(a+c-b) \leq c(a+b-c)$$

$$\Leftrightarrow \cancel{ab} + ac - a^2 \leq \cancel{ba} + bc - b^2$$

$$\Leftrightarrow ac - a^2 - bc + b^2 \leq 0$$

$$\Leftrightarrow a\underbrace{(c-a)}_{\leq 0} + b\underbrace{(c-b)}_{\leq 0} \leq 0$$

it is true.

Similarly right inequality

By rearrangement inequality:-

$$\underbrace{a^2(b+c-a) + b^2(a+c-b) + c^2(a+b-c)}_{S} \leq ba(b+c-a) + cb(c+a-b) + ac(a+b-c) \quad (2)$$

$$S \leq ca(b+c-a) + ab(c+a-b) + bc(a+b-c) \quad (1)$$

(1)+(2)

$$2S \leq 6abc$$

$$\Rightarrow S \leq 3abc$$